Floquet Theory

Consider the linear periodic system as follows.

$$\dot{x} = A(t)x, \quad A(t+p) = A(t), \quad p > 0,$$

where $A(t) \in C(R)$.

Lemma 8.4 If C is a $n \times n$ matrix with $\det C \neq 0$, then, there exists a $n \times n$ (complex) matrix B such that $e^B = C$.

Proof: For any matrix C, there exists an invertible matrix P, s.t. $P^{-1}CP = J$, where J is a Jordan matrix.

If $e^B = C$, then, $e^{P^{-1}BP} = P^{-1}e^B P = P^{-1}CP = J$**.** Therefore, it is suffice to prove the result when C is in a canonical form.

Suppose that $C = diag(C_1, \cdots, C_s)$, $C_j = \lambda_j I_j + N_j$, where N_j is nilpotent, that is,

$$N_j = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \text{ with } N_j^{n_j} = O.$$

Since C is invertible for each $\lambda_j \neq 0$.

If we can show that for each C_j, there exists B_j s.t. $C_j = e^{B_j} \Rightarrow C = e^B$.

Since $C_j = \lambda_j (I_j + \dfrac{N_j}{\lambda_j})$, using the expansion of $\ln(1+x) = \sum_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{k} x^k$, $|x| < 1$, we have

$$B_j = \ln C_j = \ln\{\lambda_j (I_j + \frac{N_j}{\lambda_j})\} = I_j \ln \lambda_j + \ln(I_j + \frac{N_j}{\lambda_j})$$

$$= I_j \ln \lambda_j + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (\frac{N^k}{\lambda^k}).$$

Since $N_j^{n_j} = O$, we actually have

$$B_j = \ln C_j = I_j \ln \lambda_j + \sum_{k=1}^{n_j - 1} \frac{(-1)^{k-1}}{k} (\frac{N^k}{\lambda^k}) = I_j \ln \lambda_j + M_j, \quad j = 1, 2, \cdots, s,$$

where $M_j = \sum_{k=1}^{n_j-1} \frac{(-1)^{k-1}}{k} (\frac{N^k}{\lambda^k})$. Therefore, we have

$$e^{B_j} = \exp\{I_j \ln \lambda_j + M_j\} = \exp\{\ln C_j\} = C_j, \quad j = 1,2,\cdots,s.$$

Let $B = diag(B_1, \cdots, B_s)$, where B_j is defined above. We have the desired result given by

$$e^B = diag(e^{B_1}, e^{B_2}, \cdots, e^{B_s}) = diag(C_1, C_2, \cdots, C_s) = C. \quad \square$$

Remark 8.16 Clearly, B is not unique since $e^{B+2\pi k i I_n} = e^B e^{2\pi k i I_n} = e^B e^{2\pi k i} I_n$ $= e^B e^{2\pi k i} = e^B$ for any integer k.

Theorem 8.6 (Floquet Theorem) If $\Phi(t)$ is a fundamental matrix solution of the periodic system $\dot{x} = A(t)x$, then so is $\Phi(t+p)$. Moreover, there exists an invertible matrix $P(t)$ with p - period such that

$$\Phi(t) = P(t)e^{Bt}.$$

Proof. Let $\Psi(t) = \Phi(t+p)$. Since $\Phi'(t) = A(t)\Phi(t)$, it follows that

$$\Psi'(t) = \Phi'(t+p) = A(t+p)\Phi(t+p) = A(t)\Psi(t),$$

Hence, $\Psi(t)$ is also a matrix solution. Since $\Phi(t)$ is invertible for all $t \in R$, so is $\Phi(t+p) \Rightarrow \Psi(t)$ is also a fundamental matrix solution. Therefore, there exists an invertible matrix C (for example, if $\Phi(t)$ satisfies $\Phi(0) = I_n$, then $C = \Phi(p)$!! Depends on solutions. It is a point of difficulty for computation) s.t.

$$\Phi(t+p) = \Phi(t)C \quad \text{for all} \quad t \in R.$$

By Lemma 8.4, there exists a matrix B such that $e^{Bp} = C$. For such a matrix B, we take $P(t) := \Phi(t)e^{-Bt}$, that is, $\Phi(t) = P(t)e^{Bt}$. Then

$$P(t+p) = \Phi(t+p)e^{-B(t+p)} = \Phi(t)Ce^{-B(t+p)} = \Phi(t)e^{-Bt} = P(t).$$

Therefore $P(t)$ is invertible for all $t \in R$ and p-periodic. This concludes the proof.

□

Remark 8.17

1) If we know $\Phi(t)$ over $[t_0, t_0+p]$, then we will know $\Phi(t)$ for all $t \in R$ by Floquet Theorem. This means that $\Phi(t)$ on $[t_0, t_0+p]$ determines $\Phi(t)$ for all $t \in R$.

Reasoning:

Suppose $\Phi(t)$ is known on $[t_0, t_0+p]$. Since $\Phi(t+p) = \Phi(t)C$, we take $C = \Phi^{-1}(t_0)\Phi(t_0+p)$ and $B = p^{-1}\ln C$. $P(t) = \Phi(t)e^{-Bt}$ is known on $[t_0, t_0+p]$. Since $P(t)$ is periodic for $t \in R$, $\Phi(t)$ is given over $t \in R$ by $\Phi(t) = P(t)e^{Bt}$.

2) If $\Phi(t)$ determines e^{Bt} (or B), then any fundamental matrix solution $\Psi(t)$ determines a similar matrix $Se^{Bp}S^{-1}$ (or SBS^{-1}).

Reasoning:

For any fundamental matrix solution $\Psi(t)$, there exists S with $\det S \neq 0$ s.t. $\Phi(t) = \Psi(t)S$. Since $\Phi(t+p) = \Phi(t)e^{Bp}$, we have

$$\Psi(t+p)\,S = \Psi(t)\,S\,e^{Bp} \quad \Rightarrow \quad \Psi(t+p) = \Psi(t)\,S\,e^{Bp}\,S^{-1} = \Psi(t)e^{SBS^{-1}p}.$$

3) For the linear periodic system, its solutions are not necessarily periodic. That is, $\Phi(t) \neq \Phi(t+p)$ in general!!! Give counter-example by yourselves.

Corollary 8.1 Under the transformation $x = P(t)\,y$, which is invertible and periodic, the periodic system $\dot{x} = A(t)x \Rightarrow$ a time-invariant system .

Proof. Suppose $P(t)$ and B defined by before and let $x = P(t)y$. Then

$$x' = P'(t)y + P(t)y' \quad \text{and} \quad x' = A(t)x = A(t)P(t)y \quad \Rightarrow \quad P'(t)y + P(t)y' = A(t)P(t)y,$$

$$\Rightarrow \quad y' = P^{-1}(t)[A(t)P(t) - P'(t)]\,y.$$

By Floquet Theorem with $P(t) = \Phi(t)e^{-Bt}$, we have

$$P'(t) = A(t)\Phi(t)e^{-Bt} + \Phi(t)e^{-Bt}(-B) = A(t)P(t) - P(t)B.$$

It follows that

$$y' = P^{-1}(t)[A(t)P(t) - P'(t)]y = P^{-1}(t)P(t)By = By\,.$$

This completes the proof. □

Remark 8.18

1) $x = P(t)y$ is called **Lyapunov transformation**. $P(t)$, which plays an important role. But it is difficult to be found explicitly since the computation of $P(t) = \Phi(t)e^{-Bt}$ depends on a fundamental matrix solution $\Phi(t)$.

2) Since $\Phi(t+p) = \Phi(t)C$ with $\det C \neq 0$, $e^{B} = C$, the eigenvalues ρ of C are called **the characteristic multipliers** of the periodic linear system. The eigenvalues λ of B are called **characteristic exponents** of the periodic linear system. $\rho = e^{\lambda p}$.

3) Since B is not unique, the characteristic exponents are not uniquely defined, but the multipliers $\{\rho\}$ are. uniquely defined (Why?) We always choose the exponents $\{\lambda\}$ as the eigenvalues of B, where B is any matrix such that $e^{Bp} = C$.

4) Since B is not unique and satisfies $e^{Bp} = C$, so B is not necessarily real.

5) B may be complex, even if C is real. However, if $A(t)$ is real (so that C is real), then, there exists a real S such that $e^{2Sp} = C^2$.

Reasoning:

Suppose $\Phi(t)$ with $\Phi(0) = I_n$, then $C = \Phi(p) = e^{Bp}$, so

$$\Phi^2(p) = e^{Bp}e^{\bar{B}p} = e^{(B+\bar{B})p}\,.$$

Let $S = \dfrac{B+\bar{B}}{2}$, then S is real s.t. $e^{2Sp} = \Phi^2(p) = C^2$.

6) Let $S(t) = \Phi(t)e^{-St}$. Then $S(t)$ is real, $2p$ -periodic.

Moreover, $x = S(t)z$ reduces the periodic system $\dot{x} = A(t)x$ into $z' = Sz$.

Reasoning:

Clearly, $S(t)$ is real since S is real, and

$$S(t+2p) = \Phi(t+2p)e^{-S(t+2p)} = \Phi(t)C^2e^{-2Sp}e^{-St} = \Phi(t)e^{-St} = S(t);$$

It is similar to obtain $\dot{z} = Sz$ under the transformation $x = S(t)z$.

- Floquet theory gives a theoretical result which reduces it into linear systems with constant coefficients. However, The Lyapunov transformation can not be computed.

- Floquet theory is very useful to study stability of a given periodic solution, noted that not equilibrium here. This is a topic of research for **dynamic systems**, or it is also named as **geometric theory of differential equations**. It is noted that this type of stability is not in Lyapunov sense.